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Electric field gradient effects in anti-plane problems of polarized ceramics

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Abstract

The equations of elastic dielectrics with electric field gradient effects are specialized to the case of anti-plane motions of polarized ceramics. A general solution is obtained in polar coordinates. Analytical solutions to the static problems of the potential field of a line source, the capacitance of a circular cylindrical ceramic shell, and the dynamic problem of the dispersion relation of plane waves are obtained to examine the electric field gradient effect. Special attention is paid to the case when the shell is thin and the waves are short.

Keywords: Piezoelectric; Gradient

1. Introduction

For elastic dielectrics there are two formulations. One uses the electric polarization vector as the independent electric constitutive variable (Toupin, 1956). The other uses the electric field vector (Tiersten, 1971). Mindlin (1968) extended the polarization formulation by allowing the stored energy density to depend on the polarization gradient, in addition to the polarization vector and the strain tensor. Mindlin's theory has received a lot of attention, e.g., Askar et al. (1970); Chowdhury and Glockner (1976, 1977); Chowdhury et al. (1979); Schwartz (1969), and Collet (1982). As is well known, gradient theories can describe size effects which are important in small scale problems. They also have important consequences in problems with singularities like concentrated sources or defects. Gradient theories are closer to microscopic theories like lattice dynamics than classical continuum theories (Mindlin, 1969). They are still applicable when the characteristic length of a problem is so small that classical continuum theories begin to fail. One of the successful applications of Mindlin's polarization gradient theory is the capacitance of a thin dielectric film (Mindlin, 1969) where a size effect was observed in experiments and the classical theory of dielectrics

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cannot describe it. The development of new technology results in thinner and thinner dielectric films and other small electronic devices. The study of these small devices presents new problems that old theories cannot describe, and the gradient effects of electric variables often play an important role in these problems. For dielectrics it is known that the electric field gradient can also be used as constitutive variables (Landau and Lifshitz, 1984). The resulting theory is called dielectrics with spatial dispersion, and is equivalent to the theory of dielectrics with electric quadrupoles (Kafadar, 1971) because electric quadrupole is the thermodynamic conjugate of the electric field gradient. Theories for elastic dielectrics with electric quadrupoles were also developed (Demiray and Eringen, 1973; Maugin, 1979, 1980; Eringen and Maugin, 1990), which provide results similar to Mindlin's polarization gradient theory in problems with singularities or scale effects. Various gradient theories can be considered as theories for weak nonlocal effects (Maugin, 1979).

In this paper we study the effect of electric field gradient in anti-plane problems of polarized ceramics. The three-dimensional equations for elastic dielectrics with electric field gradient are summarized in Section 2. Equations for the two-dimensional anti-plane case are given in Section 3. A general solution in polar coordinates is obtained in Section 4. Static problems of the potential field of a line source and the capacitance of a circular cylindrical ceramic shell are analyzed in Sections 5 and 6, respectively. Section 7 is on the dynamic problem of propagation and dispersion of plane waves. Finally, some conclusions are drawn in Section 8.

2. Elastic dielectrics with electric field gradient

We summarize the theory of elastic dielectrics with electric field gradient below. Consider the following functional over a volume V bounded by a closed surface S with unit exterior normal $\mathbf{n}$ (Yang, 1997a)

$$\Gamma(u_i,\phi)=\int_V\left[W(S_{ij},E_i,E_{i,j})-\frac{1}{2}\varepsilon_0E_iE_i-f_iu_i+q\phi\right]\mathrm{d}V-\int_S\left(\bar{t}_iu_i+\bar{d}\phi+\bar{\pi}\frac{\partial\phi}{\partial\mathbf{n}}\right)\mathrm{d}S, \tag{1}$$

where u_i is the mechanical displacement vector, ϕ the electric potential, S_{ij} the strain tensor, ε_0 the electric permittivity of free space, E_i the electric field vector, f_i the body force, q the body free charge density, and $\bar{t}_i$ the surface traction vector. $\bar{d}$ is related to surface free charge. $\bar{\pi}$ is physically more subtle and mathematically its presence is variationally consistent. Summation convention for repeated tensor indices and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index are used. In Eq. (1)

$$W(S_{ij},E_i,E_{i,j})=H(S_{ij},E_i)-\varepsilon_0\gamma_{ijk}E_iE_{j,k}-\frac{1}{2}\varepsilon_0\alpha_{ijkl}E_{i,j}E_{k,l}, \tag{2}$$

where H is the usual electric enthalpy function of piezoelectric materials. For linear materials H can be written in the following quadratic form (Tiersten, 1969)

$$H(S_{ij},E_i)=\frac{1}{2}c_{ijkl}S_{ij}S_{kl}-\frac{1}{2}\varepsilon_0\chi_{ij}E_iE_j-e_{ikl}E_iS_{kl}, \tag{3}$$

where c_{ijkl} are the elastic constants, e_{ijk} the piezoelectric constants, and χ_{ij} the electric susceptibility tensor. γ_{ijk} and α_{ijkl} are new material constants due to the introduction of the electric field gradient in the energy density function. γ_{ijk} has the dimension of length. α_{ijkl} has the dimension of (length)^2. Physically they may be related to characteristic lengths of atomic or microstructural interactions of the material. Since $E_{i,j}=E_{j,i}$, α_{ijkl} has the same structure as c_{ijkl} as required by crystal symmetry, and γ_{ijk} has the same structure as e_{ijk}. For W to be negative definite in the case of pure electric phenomena without mechanical fields, we require α_{ijkl} to be positive definite like χ_{ij}.

With the following constraints

$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \tag{4}$$

from the variational functional in Eq. (1), for independent variations of u_i and ϕ in V, we have

$$\begin{aligned} &T_{ji,j} + f_i = 0, \\ &D_{i,i} = q, \end{aligned} \tag{5}$$

where we have denoted

$$\begin{aligned} T_{ij} &= \frac{\partial W}{\partial S_{ij}} = c_{ijkl}S_{kl} - e_{kij}E_k, \\ D_i &= \varepsilon_0 E_i + P_i = \varepsilon_{ij}E_j + e_{ikl}S_{kl} - \varepsilon_0(\gamma_{kij} - \gamma_{ijk})E_{j,k} - \varepsilon_0\alpha_{ijkl}E_{k,lj}, \\ P_i &= \Pi_i - \Pi_{ij,j} = \varepsilon_0\chi_{ij}E_j + e_{ikl}S_{kl} - \varepsilon_0(\gamma_{kij} - \gamma_{ijk})E_{j,k} - \varepsilon_0\alpha_{ijkl}E_{k,lj}, \\ \Pi_i &= -\frac{\partial W}{\partial E_i} = e_{ikl}S_{kl} + \varepsilon_0\chi_{ij}E_j + \varepsilon_0\gamma_{ijk}E_{j,k}, \\ \Pi_{ij} &= -\frac{\partial W}{\partial E_{i,j}} = \varepsilon_0\gamma_{kij}E_k + \varepsilon_0\alpha_{ijkl}E_{k,l}. \end{aligned} \tag{6}$$

In Eq. (6) $\varepsilon_{ij} = \varepsilon_0(\delta_{ij}\chi_{ij})$. Π_i and Π_{ij} are related to electric dipole and quadrupole densities (Eringen and Maugin, 1990). The functional in Eq. (1) also implies the following as possible forms of boundary conditions on S:

$$\begin{aligned} &T_{ji}n_j = \bar{t}_i \quad \text{or} \quad \delta u_i = 0, \\ &\int_S [(D_i n_i - \bar{d})\delta\phi + \Pi_{ij}n_j(\nabla_s\delta\phi)_i]\,\mathrm{d}S = 0, \\ &\Pi_{ij}n_j n_i = \bar{\pi} \quad \text{or} \quad \delta\left(\frac{\partial\phi}{\partial\mathbf{n}}\right) = 0, \end{aligned} \tag{7}$$

where ∇_s is the surface gradient operator. One obvious possibility of Eq. $(7)_2$ is $\delta\phi = 0$ on S. Other possibilities will be dealt with in specific problems. With substitutions from Eq. (6), Eq. (5) can be written as four equations for u_i and ϕ

$$\begin{aligned} &c_{ijkl}u_{k,lj} + e_{kij}\phi_{,kj} + f_i = \rho\ddot{u}_i, \\ &e_{ikl}u_{k,li} - \varepsilon_{ij}\phi_{,ij} + \varepsilon_0\alpha_{ijkl}\phi_{,ijkl} = q, \end{aligned} \tag{8}$$

where ρ is mass density. A superimposed dot represents a time derivative. In Eq. (8), to include dynamic problems, we have added the acceleration term. When $\alpha \to 0$ Eqs. (8) reduce to the classical theory of piezoelectricity.

3. Anti-plane problems of polarized ceramics

For ceramics poled in the x_3 direction the material tensors in Eq. (8) are the same as those of crystals of 6 mm symmetry and are represented by the following matrices under the compact matrix notation (Tiersten, 1969)

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \quad \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}, \tag{9}$$

where $c_{66} = (c_{11} - c_{12})/2$, and the superscript T is for matrix transpose. We consider anti-plane motions with

$$\begin{aligned} &u_1 = u_2 = 0, \quad u_3 = u(x_1, x_2, t), \\ &\phi = \phi(x_1, x_2, t). \end{aligned} \tag{10}$$

The nonvanishing strain and electric field components are

$$\begin{Bmatrix} S_5 \\ S_4 \end{Bmatrix} = \nabla u, \quad \begin{Bmatrix} E_1 \\ E_2 \end{Bmatrix} = -\nabla \phi, \tag{11}$$

where ∇ is the two-dimensional gradient operator. The nontrivial components of T_{ij} and D_i are

$$\begin{aligned} &\begin{Bmatrix} T_5 \\ T_4 \end{Bmatrix} = c\nabla u + e\nabla \phi, \\ &\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = e\nabla u - \varepsilon \nabla \phi + \varepsilon_0 \alpha \nabla(\nabla^2 \phi), \\ &D_3 = -\varepsilon_0(\gamma_{31} - \gamma_{15})\nabla^2 \phi, \end{aligned} \tag{12}$$

where ∇^2 is the two-dimensional Laplacian, $c = c_{44}$, $e = e_{15}$, $\varepsilon = \varepsilon_{11}$ and $\alpha = \alpha_{11}$. The nontrivial ones of Eq. (8) take the following form

$$\begin{aligned} &c\nabla^2 u + e\nabla^2 \phi + f = \rho \ddot{u}, \\ &e\nabla^2 u - \varepsilon \nabla^2 \phi + \varepsilon_0 \alpha \nabla^2 \nabla^2 \phi = q, \end{aligned} \tag{13}$$

where $f = f_3$ Eq. (13) can be partially decoupled into the following one-way coupled system:

$$\begin{aligned} &\bar{c}\nabla^2 u + f + \frac{\varepsilon_0}{\varepsilon} \alpha \nabla^2 (\rho \ddot{u} - c\nabla^2 u - f) = \rho \ddot{u} + \frac{e}{\varepsilon} q, \\ &\nabla^2 \phi = \frac{1}{e}(\rho \ddot{u} - c\nabla^2 u - f). \end{aligned} \tag{14}$$

For static problems Eq. (13) can also be decoupled into

$$\begin{aligned} &-\bar{\varepsilon}\nabla^2 \phi + \varepsilon_0 \alpha \nabla^2 \nabla^2 \phi = q + \frac{e}{c} f, \\ &\nabla^2 u = -\frac{1}{c}(e\nabla^2 \phi + f). \end{aligned} \tag{15}$$

In Eqs. (14) and (15)

$$\bar{c} = c(1 + k^2), \bar{\varepsilon} = \varepsilon(1 + k^2), k^2 = e^2/(\varepsilon c) \tag{16}$$

where k is a dimensionless number (the electromechanical coupling factor).

4. A general solution in polar coordinates

We consider static problems with $q = 0$ and $f = 0$. Let

$$F = \nabla^2 \phi, \quad \beta^2 = \bar{\varepsilon}/\varepsilon_0 \alpha. \tag{17}$$

Eq. $(15)_1$ becomes

$$\nabla^2 F - \beta^2 F = 0. \tag{18}$$

In a polar coordinate system defined by $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, by separation of variables, the general solution for F periodic in θ can be found as

$$F = \sum_{n=0}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)[c_n I_n(\beta r) + d_n K_n(\beta r)], \tag{19}$$

where a_n, b_n, c_n and d_n are undetermined constants, and I_n and K_n are modified Bessel functions of order n of the first- and second-kind. Then from Eq. $(17)_1$ the general solution for ϕ is

$$\begin{aligned}\phi = {} & a_0 \left[g_0 + h_0 \ln r + \frac{c_0}{\beta^2} I_0(\beta r) + \frac{d_0}{\beta^2} K_0(\beta r) \right] \\ & + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \left[g_n r^n + h_n r^{-n} + \frac{c_n}{\beta^2} I_n(\beta r) + \frac{d_n}{\beta^2} K_n(\beta r) \right],\end{aligned} \tag{20}$$

where g_n and h_n are undetermined constants. Once ϕ is known, from Eq. $(15)_2$ u is given by

$$\begin{aligned}u = {} & a_0 \left[l_0 + p_0 \ln r - \frac{e}{c}\frac{c_0}{\beta^2} I_0(\beta r) - \frac{e}{c}\frac{d_0}{\beta^2} K_0(\beta r) \right] \\ & + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \left[l_n r^n + p_n r^{-n} - \frac{e}{c}\frac{c_n}{\beta^2} I_n(\beta r) - \frac{e}{c}\frac{d_n}{\beta^2} K_n(\beta r) \right],\end{aligned} \tag{21}$$

where l_n and p_n are undetermined constants.

5. Potential field of a line source

We now consider the potential field of a line charge Q at the origin. The above general solution is applicable and it is also simple to integrate the equation directly. We need to solve Eq. $(15)_1$ with a concentrated source term

$$-\bar{\varepsilon}\nabla^2\phi + \varepsilon_0\alpha\nabla^2\nabla^2\phi = Q\delta(x_1, x_2). \tag{22}$$

Eq. (22) can be rewritten as

$$(-\bar{\varepsilon} + \varepsilon_0\alpha\nabla^2)\nabla^2\phi = Q\delta(x_1, x_2). \tag{23}$$

Therefore $\nabla^2\phi$ is the fundamental solution of the differential operator in Eq. (23), which is known (Zauderer, 1983). Hence

$$\nabla^2\phi = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = -\frac{Q}{2\pi\varepsilon_0\alpha}K_0(\beta r). \tag{24}$$

Since

$$xK_0(x) = -\frac{\mathrm{d}}{\mathrm{d}x}[xK_1(x)], \quad K_1(x) = -\frac{\mathrm{d}}{\mathrm{d}x}[K_0(x)]. \tag{25}$$

Integrating Eq. (24) twice we obtain

$$\phi = -\frac{Q}{2\pi\bar{\varepsilon}}[\ln r + K_0(\beta r)], \tag{26}$$

where the $\ln r$ term is the classical solution. Since

$$\begin{aligned} &K_0(x) \to -\ln x, \quad x \to 0, \\ &K_0(x) \to \left(\frac{\pi}{2x}\right)^{1/2} \mathrm{e}^{-x}, \quad x \to \infty, \end{aligned} \tag{27}$$

we have

$$\begin{aligned} &\phi \to \frac{Q}{2\pi\bar{\varepsilon}} \ln \beta = \frac{Q}{4\pi\bar{\varepsilon}} \ln \frac{\bar{\varepsilon}}{\varepsilon_0 \alpha}, \quad r \to 0, \\ &\phi \to -\frac{Q}{2\pi\bar{\varepsilon}} \ln r, \quad r \to \infty. \end{aligned} \tag{28}$$

Therefore for far field ϕ approaches the classical solution. At the origin the exact value of ϕ is not important because an arbitrary constant in ϕ is immaterial. The important thing is that at the source point ϕ is not singular. This is fundamentally different from the classical solution. When α approaches zero, $\beta \to \infty$ from Eqs. (17) and (26) reduces to the classical result. The potential field is plotted in Fig. 1 to show its qualitative behavior which is as expected. The curve with the smaller value of α or the larger value of β is closer to the classical solution.

6. Capacitance of a circular cylindrical shell

Experiments show that the capacitance of a thin dielectric plate is smaller than the prediction of the classical theory. This difference was explained by Mindlin using both lattice dynamics and his polarization gradient theory (Mindlin, 1969). In this section we show that a similar phenomenon is predicted for a circular cylindrical shell capacitor of polarized ceramics. Consider a circular cylindrical shell of inner radius

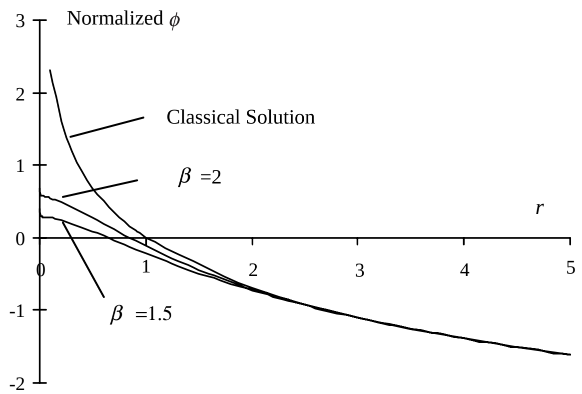

Fig. 1. Normalized potential field $(-2\pi\bar{\varepsilon}\phi/Q)$ of a line source.

R_1 and outer radius R_2 (Fig. 2). The inner and outer surfaces are traction free and are electroded, with electrodes shown by the thick lines in the figure. A voltage V is applied across the thickness. We list the solution from the classical theory of piezoelectricity below for comparison

$$\phi = \frac{V}{\ln R_2/R_1}\ln r, \quad u = -\frac{e}{c}\frac{V}{\ln R_2/R_1}\ln r, \quad T_{rz} = 0, \quad D_r = -\bar{\varepsilon}\frac{V}{\ln R_2/R_1}\frac{1}{r}, \quad C_0 = \frac{2\pi\bar{\varepsilon}}{\ln R_2/R_1}, \tag{29}$$

where we have denoted $\chi_{11} = \chi$, and introduced $\bar{\chi}$ such that $\bar{\varepsilon} = \varepsilon_0(1+\bar{\chi})$. $\bar{\chi}$ is the effective electric susceptibility including the effect of piezoelectric coupling and is determined by $(1+\bar{\chi}) = (1+\chi)(1+k^2)$. C_0 is the capacitance per unit length of the shell.

We now seek the gradient solution. Since the problem is axi-symmetric, terms corresponding to $n = 0$ in the general solution are sufficient. From Eqs. (20) and (21) we have

$$\begin{aligned} \phi &= g_0 + h_0 \ln r + \frac{c_0}{\beta^2} I_0(\beta r) + \frac{d_0}{\beta^2} K_0(\beta r), \\ u &= l_0 + p_0 \ln r - \frac{e}{c}\frac{c_0}{\beta^2} I_0(\beta r) - \frac{e}{c}\frac{d_0}{\beta^2} K_0(\beta r), \end{aligned} \tag{30}$$

where we have dropped a_0 which is immaterial. Corresponding to Eq. (30) we have

$$\begin{aligned} S_{rz} &= u_{,r} = \frac{p_0}{r} - \frac{e}{c\beta} c_0 I_1(\beta r) + \frac{e}{c\beta} d_0 K_1(\beta r), \\ E_r &= -\phi_{,r} = -\frac{h_0}{r} - \frac{1}{\beta} c_0 I_1(\beta r) + \frac{1}{\beta} d_0 K_1(\beta r), \\ T_{rz} &= (cp_0 + eh_0)\frac{1}{r}, \\ D_r &= (ep_0 - \varepsilon h_0)\frac{1}{r}. \end{aligned} \tag{31}$$

We note that the strain and electric fields differ from the classical theory by terms of modified Bessel functions. For boundary conditions we have

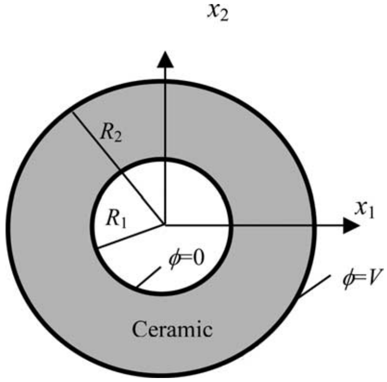

Fig. 2. A circular cylindrical ceramic shell as a capacitor.

$$T_{rz}(R_2) = (cp_0 + eh_0)\frac{1}{R_2} = 0,$$
$$\phi(R_2) - \phi(R_1) = h_0 \ln R_2/R_1 + \frac{c_0}{\beta^2}[I_0(\beta R_2) - I_0(\beta R_1)] + \frac{d_0}{\beta^2}[K_0(\beta R_2) - K_0(\beta R_1)] = V,$$
$$-E_r(R_2) = \left.\frac{\partial\phi}{\partial\mathbf{n}}\right|_{R_2} = \frac{h_0}{R_2} + \frac{1}{\beta}c_0 I_1(\beta R_2) - \frac{1}{\beta}d_0 K_1(\beta R_2) = \lambda\frac{1}{R_2}\frac{V}{\ln R_2/R_1},$$
$$-E_r(R_1) = \left.\frac{\partial\phi}{\partial\mathbf{n}}\right|_{R_1} = \frac{h_0}{R_1} + \frac{1}{\beta}c_0 I_1(\beta R_1) - \frac{1}{\beta}d_0 K_1(\beta R_1) = \lambda\frac{1}{R_1}\frac{V}{\ln R_2/R_1}. \quad (32)$$

Corresponding to Eq. $(7)_{1,2}$, the first two boundary conditions on traction and voltage in Eq. $(32)_{1,2}$ are formally classical except the terms of the modified Bessel functions. The surface gradient term in Eq. $(7)_2$ disappears due to the symmetry in this problem. Since the order of the equation in Eq. $(15)_1$ is higher than the Laplace equation in the classical theory, new boundary conditions are needed. According to Eq. $(7)_3$, the additional boundary conditions may be either on $\partial\phi/\partial\mathbf{n}$ or $\Pi_{ij}n_in_j$. Motivated by Mindlin (1969), what we have in Eq. $(32)_{3,4}$ is on $\partial\phi/\partial\mathbf{n}$, where, following (Mindlin, 1969), we have introduced a parameter λ and when $\lambda = 1$ the right hand sides of Eq. $(32)_{3,4}$ represent the values of $-E_r(R_2)$ and $-E_r(R_1)$ of the classical solution. Mindlin concluded based on physical reasoning (Mindlin, 1969) that the boundary normal values of **P** in a plate capacitor from his polarization gradient theory should be smaller than the values from the classical theory. Since we are using electric field gradient, Mindlin's argument should translate into that the boundary normal values of **E** be larger than the classical values. Therefore we require that $\lambda > 1$. This is also consistent with the results on the electric field from a nonlocal analysis of a thin plate capacitor (Yang, 1997b). According to (Mindlin, 1969), λ depends on the physical nature of the electrode-dielectric interface. We note that $T_{rz}(R_1) = 0$ does not provide an independent boundary condition because Eq. $(32)_1$ implies that $T_{rz} = 0$ everywhere. Eq. (32) represents four equations for h_0, p_0, c_0 and d_0. They are solved on a computer. The capacitance of the shell per unit length is determined by

$$C = -2\pi R_2 D_r(R_2)/V = -2\pi(ep_0 - \varepsilon h_0)/V. \quad (33)$$

Normalized capacitance C/C_0 versus $(R_2 - R_1)/\sqrt{\alpha}$ is plotted in Fig. 3. It is seen that when $(R_2 - R_1)/\sqrt{\alpha}$ is large the gradient and classical solutions agree, but when $(R_2 - R_1)/\sqrt{\alpha}$ is small the gradient solution is smaller. This was observed experimentally for plate capacitors (Mindlin, 1969) and it is natural to expect the same for a shell capacitor.

Electric field distribution along the shell thickness is shown in Fig. 4. Except near the surfaces of the shell, the gradient and the classical solutions show the same electric field. Near the surfaces the gradient

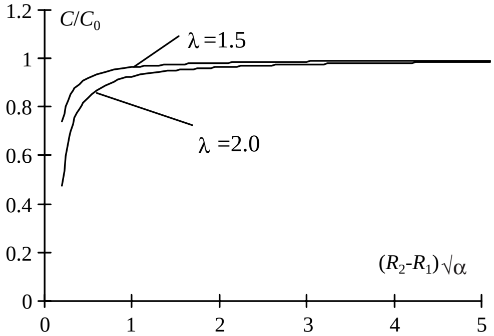

Fig. 3. Normalized capacitance versus shell thickness in a circular cylindrical shell ($R_1 = 10\sqrt{\alpha}$).

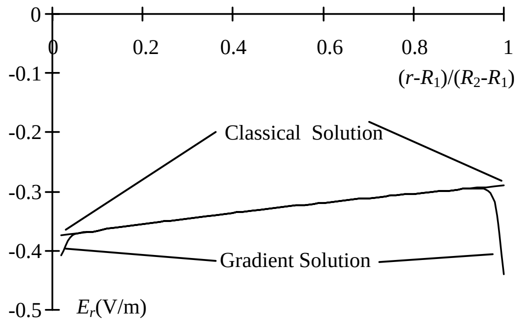

Fig. 4. Electric field distribution along the shell thickness in a circular cylindrical shell ($R_1 = 10\sqrt{\alpha}$, $R_2 = 13\sqrt{\alpha}$, $\lambda = 1.5$, $V = 10^{-6}$ V).

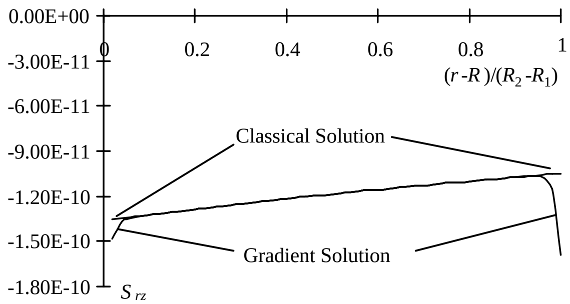

Fig. 5. Strain field distribution along the shell thickness in a circular cylindrical shell ($R_1 = 10\sqrt{\alpha}$, $R_2 = \sqrt{\alpha}$, $\lambda = 1.5$, $V = 10^{-6}$ V).

solution predicts larger electric fields. This type of boundary behavior is typical for a gradient theory and is also seen in the polarization gradient solution for a plate in (Mindlin, 1969), and the nonlocal solution in (Yang, 1997b) where there are no additional boundary conditions and the parameter λ is not needed.

The strain field distribution along the shell thickness is shown in Fig. 5, which is very similar to the electric field. The localized electromechanical fields near the shell surfaces may have implications on the strength of the shell.

Numerical results also show that

$$\frac{D_r^{\mathrm{Gradient}}}{D_r^{\mathrm{Classical}}} = 0.9925, \tag{34}$$

which supports our reasoning that a smaller **P** in the polarization gradient theory translates into a larger **E** in the electric field gradient theory.

7. Propagation of plane waves

Consider the propagation of the following plane wave

$$u = \mathrm{e}^{\mathrm{i}(\xi x_1 - \omega t)}, \tag{35}$$

where ξ is wave number and ω is frequency. Substitution of Eq. (35) into the homogeneous form of Eq. $(14)_1$ ($f = 0$, $q = 0$) yields the following dispersion relation

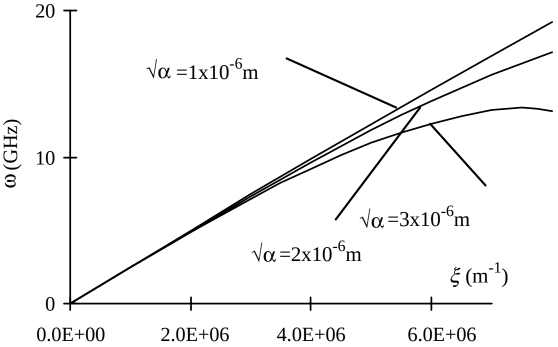

Fig. 6. Dispersion curves of plane waves.

$$\omega^2 = \frac{c}{\rho}\xi^2\left[1 + \frac{k^2}{1 + \frac{\varepsilon_0}{\varepsilon}\alpha\xi^2}\right]. \tag{36}$$

Different from the plane waves in linear piezoelectricity, Eq. (30) shows that the waves are dispersive, and the dispersion is caused by electric field gradient through electromechanical coupling. The dispersion disappears when $k = 0$, i.e., when there is no electromechanical coupling. We note that the dispersion is more pronounced when $\xi/\sqrt{\alpha}$ is not small, or when the wavelength $2\pi\xi$ is not large when compared to the microscopic characteristic length $\sqrt{\alpha}$. When $\xi/\sqrt{\alpha}$ just begins to show its effect, Eq. (36) can be approximated as

$$\omega^2 \cong \frac{\bar{c}}{\rho}\xi^2\left[1 - \frac{k^2}{1+k^2}\frac{\varepsilon_0}{\varepsilon}\alpha\xi^2\right] \tag{37}$$

Eq. (37) shows that given a wave number the electric field gradient lowers the wave frequency. In this sense, as pointed by Mindlin (1972), the first strain gradient theory is fundamentally flawed because it predicts an increase of wave frequency which is inconsistent with lattice dynamics. The second strain gradient needs to be included to correct this inconsistency. As a numerical example we consider polarized ceramics PZT-7A with the following material constants (Jaffe and Berlincourt, 1965)

$$\begin{aligned} &\rho = 7500\ \text{kg/m}^3, \quad c_{11} = 148, \quad c_{33} = 131, \quad c_{12} = 76.2, \quad c_{13} = 74.2, \\ &c_{44} = 25.4, \quad c_{66} = 35.9\ \text{GPa}, \quad e_{15} = 9.2, \quad e_{31} = -2.1, \quad e_{33} = 9.5\ \text{C/m}^2, \\ &\varepsilon_{11} = 460\varepsilon_0, \quad \varepsilon_{33} = 235\varepsilon_0, \quad \varepsilon_0 = 8.85 \times 10^{-12}\ \text{F/m}. \end{aligned} \tag{38}$$

For polarized ferroelectric ceramics the grain size, which can be taken as the microscopic characteristic length $\sqrt{\alpha}$, is at sub-micron range. We plot Eq. (36) in Fig. 6 for different values of $\sqrt{\alpha}$. It can be seen that larger values of $\sqrt{\alpha}$ yields more dispersion, as expected.

8. Conclusion

For anti-plane problems the equations for elastic dielectrics with electric field gradient reduce to two relatively simple equations which allow a general solution in polar coordinates. In statics, different from the classical theory of piezoelectricity, for a concentrated source the potential field is not singular. The capacitance of a thin circular cylindrical ceramic shell is predicted to be smaller than the classical theory. The strain and electric fields differ from the classical results mainly near the surfaces of the shell. For dynamic problems the electric field gradient causes plane waves to become dispersive through electromechanical coupling. The dispersion is more pronounced when the waves are short in the sense that the wavelength approaches a microscopic characteristic length. The electric field gradient is expected to have similar effects in other small scale problems or problems with singularities.

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